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BIGRADED STRUCTURES AND THE DEPTH OF BLOW-UP ALGEBRAS

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Abstract. Let R be a Cohen-Macaulay local ring, and let $I \subset R$ be an ideal with minimal reduction J. In this paper we attach to the pair I, J a non-standard bigraded module $\Sigma^{I,J}$. The study of the bigraded Hilbert function of $\Sigma^{I,J}$ allows us to prove a improved version of Wang's conjecture and a weak version of Sally's conjecture, both on the depth of the associated graded ring $gr_I(R)$. The module $\Sigma^{I,J}$ can be considered as a refinement of the Sally's module previously introduced by W. Vasconcelos.

Introduction

Let $(R, \mathbf{m}, \mathbf{k})$ be a d-dimensional Cohen-Macaulay local ring. Let I be an $\mathbf{m}-$primary ideal of R with minimal reduction J. One of the major problems in commutative algebra is to estimate the depth of the associated graded ring $gr_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ and the Rees algebra $\mathcal{R}(I) = \bigoplus_{n\geq 0} I^n t^n$ for ideals I having good properties. Attached to the pair I, J we can consider the integers

$$\Delta(I,J) = \sum_{p\geq 1} length_R\left(\frac{I^{p+1}\cap J}{I^p J}\right) \quad , \quad \Lambda(I,J) = \sum_{p\geq 0} length_R\left(\frac{I^{p+1}}{JI^p}\right),$$

$\Delta_p(I,J) = length_R(I^{p+1}\cap J/JI^p)$, and $\Lambda_p(I,J) = length_R(I^{p+1}/JI^p)$, for $p \geq 0$.

Related to these integers there are some results and conjectures on the depth of $gr_I(R)$ that we next review. Valabrega and Valla proved that if $\Delta(I,J) = 0$ then $gr_I(R)$ is Cohen-Macaulay, [16]. Based in this result Guerrieri proposed the following conjecture:

Conjecture 0.1 (Guerrieri, [6]). *Let I be an $\mathbf{m}-$primary ideal of R with minimal reduction J. Then*

$$depth(gr_I(R)) \geq d - \Delta(I,J).$$

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Guerrieri proved the case $\Delta(I,J)=1$ and some partial cases for $\Delta(I,J)=2$, [6]. Wang proved the case $\Delta(I,J)=2$ without any restriction, [21].

Guerrieri in her thesis asked if the conditions $\Delta_p(I,J)\leq 1$, $p\geq 1$, implies that $depth(gr_I(R))\geq d-1$, [5], Question 2.23. Wang in [23], Example 3.13, gave a counterexample to Guerrieri's question and asked if this question has an affirmative answer assuming that R is a regular local ring.

Huckaba and Marley proved that $e_1(I)\leq \Lambda(I,J)$ and if the equality holds then $depth(gr_I(R))\geq d-1$, [11]. Hence we can consider the non-negative integer $\delta(I,J)=\Lambda(I,J)-e_1(I)\geq 0$. Wang showed that $\delta(I,J)\leq \Delta(I,J)$ and that Guerrieri's conjecture is implied by the following one, [21],

Conjecture 0.2 (Wang)**.** *Let I be an $\mathbf{m}-$primary ideal of R with minimal reduction J. Then*

$$depth(gr_I(R))\geq d-1-\delta(I,J).$$

Huckaba proved the conjecture in the case $\delta(I,J)=0$, [9], [11]. If $\delta(I,J)=1$ Wang proved the conjecture and Polini gave a simpler proof, [21], [13]. For $\delta(I,J)=2$ Rossi and Guerrieri proved Wang's conjecture assuming that R/I is Gorenstein, [7]. Wang gave a counterexample to the conjecture for $d=6$, [22].

In the main result of this paper we prove a refined version of Wang's conjecture, Theorem 3.3. We naturally decompose the integer $\delta(I,J)=\sum_{p\geq 0}\delta_p(I,J)$ as a finite sum of non-negative integers $\delta_p(I,J)$, with $\Delta_p(I,J)\geq\delta_p(I,J)\geq 0$, see section two. Let us consider the maximum, say $\bar{\delta}(I,J)$, of the integers $\delta_p(I,J)$ for $p\geq 0$.

Theorem 3.3. *Assume that $\bar{\delta}(I,J)\leq 1$. Then*

$$depth(\mathcal{R}(I))\geq d-\bar{\delta}(I,J)$$

and $depth(gr_I(R))\geq d-1-\bar{\delta}(I,J)$.

There is another conjecture that considers some conditions on the modules I^{p+1}/JI^p and $I^p\cap J/I^{p-1}J$, it is Sally's conjecture:

Conjecture 0.3 (Sally)**.** *Let I be an $\mathbf{m}-$primary ideal of R with minimal reduction J. If $I^n\cap J=I^{n-1}J$ for $n=2,\dots,t$ and $length(\frac{I^{t+1}}{JI^t})=\epsilon\leq$ $Min\{1,d-1\}$ then it holds*

$$d-\epsilon\leq depth(gr_I(R))\leq d.$$

This conjecture was proved by Corso-Polini-Vaz-Pinto, Elias, and Rossi, [3], [4], [15].

The aim of this work is to introduce a non-standard bigraded module $\Sigma^{I,J}$ in order to study the depth of the associated graded ring $gr_I(R)$ and the Rees algebra $\mathcal{R}(I)$ of I. A secondary purpose is to present a unified framework where several results and objects appearing in the papers on the above conjectures can be studied, Remark 2.9. The key tool of this paper is the Hilbert function of non-standard bigraded modules.

The first section is mainly devoted to recall some preliminary results on Sally module and the Hilbert function of non-standard bigraded modules.

In Section two we introduce a non-standard bigraded module $\Sigma^{I,J}$ naturally attached to I and a minimal reduction J of I, this module can be considered as a refinement of the Sally module previously introduced by W. Vasconcelos. From a natural presentation of $\Sigma^{I,J}$ we define two bigraded modules $K^{I,J}$ and $\mathcal{M}^{I,J}$, and we consider some diagonal submodules of them: $\Sigma^{I,J}_{[p]}$ and $K^{I,J}_{[p]}$.

For all $p \geq 0$ we consider the integer $\delta_p(I,J) = e_0(K^{I,J}_{[p]})$. In Proposition 2.10 we prove following inequalities

$$\Delta_p(I,J) \geq \delta_p(I,J) = \Lambda_p(I,J) - e_0(\Sigma^{I,J}_{[p]}) \geq 0.$$

Summing up these inequalities with respect p we recover the inequalities, [21], [11],

$$\Delta(I,J) \geq \delta(I,J) = \Lambda(I,J) - e_1(I) \geq 0.$$

In particular we decompose the first Hilbert coefficient $e_1(I)$ of I as a sum of the multiplicities $e_0(\Sigma^{I,J}_{[p]})$ when p ranges the set of non-negative integers:

$$e_1(I) = \sum_{p \geq 0} e_0(\Sigma^{I,J}_{[p]}).$$

Section three is devoted to prove a refined version of Wang's conjecture by considering some special configurations of the set $\{\delta_p(I,J)\}_{p\geq 0}$ instead of $\delta(I,J) = \sum_{p\geq 0} \delta_p(I,J)$, Theorem 3.3. This version allows us to improve the bound for the *depth* in Guerrieri's question, Proposition 3.5 (see [5], Question 2.23). As a by-product we recover the known cases of Wang's conjecture, Corollary 3.6, and we prove a weak version of Sally's conjecture, Corollary 3.7. An essential point of this section is based on generalize a part of the work of Polini in [13] following her ideas.

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1. Preliminaries

Let $(R, \mathbf{m})$ be a local ring of dimension $d > 0$, and let I be an $\mathbf{m}$-primary ideal of R. Without loss of generality we may assume that the residue field $\mathbf{k} = R/\mathbf{m}$ is infinite. We set $I^i = 0$ for $i < 0$, and $I^i = R$ for $i = 0$.

The Rees algebra of R associated to I is the $R-$algebra

$$\mathcal{R}(I) = \bigoplus_{n\geq 0} I^n t^n$$

and the associated graded ring of R with respect to I is

$$gr_I(R) = \bigoplus_{n\geq 0} \frac{I^n}{I^{n+1}} t^n.$$

An ideal $J \subseteq I$ is said to be a reduction of I if there exists an integer $r \geq 0$ such that $I^{r+1} = JI^r$. J is a minimal reduction of I if J is a reduction of I an J itself does not contain any proper reduction. If J is a minimal reduction of I, the reduction number of I with respect to J is the least integer $r_J(I)$ such that $I^{r+1} = JI^r$ for all $r \geq r_J(I)$. The reduction number $r(I)$ of I is defined as lowest integer $r_J(I)$ where J is a minimal reduction of I.

We denote by $h_I^0(n) = length_R(I^n/I^{n+1})$ the $0-th$ Hilbert function of I. The higher Hilbert functions of I are defined by, $i \geq 0$,

$$h_I^{i+1}(n) = \sum_{j=0}^{n} h_I^i(j).$$

It is well known that there exist integers $e_j(I) \in \mathbb{Z}$ such that if we write

$$p_I^i(X) = \sum_{j=0}^{d+i-1} (-1)^j e_j(I) \binom{X+d+i-j-1}{d+i-j-1}$$

then p_I^i is the $i-$th Hilbert polynomial of I, i.e. $h_I^i(n) = p_I^i(n)$ for $n \gg 0$. The integer $e_i(I)$ is the $i-$th Hilbert coefficient of I. In this paper we set $h_I = h_I^0$ and $p_I = p_I^0$. We denote by $pn(I)$ the postulation number of h_I, i.e. the least integer t such that $h_I(t+n) = p_I(t+n)$ for all $n \geq 0$.

Let I an $\mathbf{m}$-primary ideal of R and J a minimal reduction of I. The Sally module of I with respect to J is the $\mathcal{R}(J)$-module

$$S_J(I) = \frac{I\mathcal{R}(I)}{I\mathcal{R}(J)} = \bigoplus_{n\geq 1} \frac{I^{n+1}}{J^n I} t^n$$

We define the Hilbert function of the Sally module $S_J(I)$ as, [17],

$$h_{S_J(I)}(n) = length_R(I^{n+1}/J^n I).$$

If $S_J(I) \neq 0$ then $dim(S_J(I)) = d$, and we can consider the Hilbert polynomial of $S_J(I)$

$$p_{S_J(I)}(n) = \sum_{i=0}^{d-1} (-1)^i s_i \binom{n+d-i-1}{d-i-1}.$$

In the next proposition we collect some known results on the Sally module.

Proposition 1.1. *Let $(R, \mathbf{m})$ a Cohen-Macaulay local ring of dimension $d > 0$. Let I be an $\mathbf{m}$-primary ideal of R and J a minimal reduction of I.*

(i) *If $S_J(I) = 0$ then $gr_I(R)$ is Cohen-Macaulay. If $S_J(I) = 0$ and $d \geq 2$, then $\mathcal{R}(I)$ is Cohen-Macaulay.*
(ii) *$depth(gr_I(R)) \geq depth(S_J(I)) - 1$.*
(iii) *If $depth(gr_I(R)) < d$, then $depth(S_J(I)) = depth(gr_I(R)) + 1$.*
(iv) *The Hilbert coefficients of $S_J(I)$ and I are related by the following equalities*

$$\begin{cases} e_0(I) = length_R(R/J) \\ e_1(I) = e_0(I) - length_R(R/I) + s_0 = s_0 + length_R(I/J) \\ e_i(I) = s_{i-1} & i = 2, \dots, d \end{cases}$$

Roberts proved the existence of Hilbert polynomials of bigraded modules over bigraded rings $\mathbf{k}[X_1, \dots, X_s]$ with variables $X_1, \dots, X_s$ of bidegrees $(1, 0)$, $(0, 1)$, and $(1, 1)$, here $\mathbf{k}$ is a field, [14], see also [8]. The results of Section 3 of [14] can be easily generalized to polynomial rings with coefficients in an Artin ring.

Let $A = C[X_0, \dots, X_s, T_0, \dots, T_u, S_1, \dots, S_v]$ be a bigraded polynomial ring over an Artin ring C in the variables $X_0, \dots, X_s, T_0, \dots, T_u$ and $S_1, \dots, S_v$. We assume that the variables X_i have bidegree $(1, 0)$, the variables T_i have bidegree $(1, 1)$, and the variables S_i have bidegree $(0, 1)$.

For a bigraded A-module M and for any $m, n \in \mathbb{Z}$, let $M_{(m,n)}$ be the piece of M of bidegree (m, n). Let $h_M(m, n)$ be the Hilbert function of M defined by the equality

$$h_M(m, n) = \sum_{i \leq n} length_A(M_{(m,i)}).$$

Theorem 1.2. *Let $A = C[X_0, \dots, X_s, T_0, \dots, T_u, S_1, \dots, S_v]$ be a bigraded polynomial ring over an Artin ring C in variables $X_0, \dots, X_s, T_0, \dots, T_u$ and $S_1, \dots, S_v$,*

where each X_i has bidegree $(1,0)$, each T_i has bidegree $(1,1)$, and each S_i has bidegree $(0,1)$. For all finitely generated bigraded A-module M, there exist a polynomial in two variables $p_M(m,n)$, and integers m_0 and n_0 such that

$$p_M(m,n) = h_M(m,n)$$

for all (m,n) with $m \geq m_0$ and $n \geq m + n_0$.

2. Bigraded Sally module

Let R be a Cohen-Macaulay local ring. Let $I = (b_1, \dots, b_\mu)$ be an $\mathbf{m}$-primary ideal of R and let $J = (a_1, \dots, a_d)$ be a minimal reduction of I. Since $Jt\mathcal{R}(I)$ is an homogeneous ideal of the graded ring $\mathcal{R}(I)$ we can consider the associated graded ring of $\mathcal{R}(I)$ with respect to the homogeneous ideal $Jt\mathcal{R}(I) = \bigoplus_{n\geq 0} JI^{n-1}t^n$

$$gr_{Jt}(\mathcal{R}(I)) = \bigoplus_{j\geq 0} \frac{(Jt\mathcal{R}(I))^j}{(Jt\mathcal{R}(I))^{j+1}} U^j.$$

This ring has a natural bigraded structure that we describe briefly. Notice that $\frac{\mathcal{R}(I)}{Jt\mathcal{R}(I)} = \bigoplus_{i\geq 0} \frac{I^i}{I^{i-1}J} t^i$ is a homomorphic image of the graded ring $R[V_1, \dots, V_\mu]$ by the degree one R-algebra homogeneous morphism

$$\sigma : R[V_1, \dots, V_\mu] \longrightarrow \frac{\mathcal{R}(I)}{Jt\mathcal{R}(I)} = \bigoplus_{i\geq 0} \frac{I^i}{I^{i-1}J} t^i$$

defined by $\sigma(V_i) = b_i t \in \frac{I}{J} t$; $R[V_1, \dots, V_\mu]$ is endowed with the standard graduation.

Let us consider the bigraded ring $B := R[V_1, \dots, V_\mu; T_1, \dots, T_d]$ with $deg(V_i) = (1,0)$ and $deg(T_i) = (1,1)$, then there exists an exact sequence of bigraded B-rings

$$0 \longrightarrow K^{I,J} \longrightarrow C^{I,J} := \frac{\mathcal{R}(I)}{Jt\mathcal{R}(I)}[T_1, \dots, T_d] \xrightarrow{\pi} gr_{Jt}(\mathcal{R}(I)) \longrightarrow 0 \tag{1}$$

with $\pi(T_i) = a_i t U$, $i = 1, \dots, d$; $K^{I,J}$ is the ideal of initial forms of $Jt\mathcal{R}(I)$. The $(i+j,j)$-graded piece of $gr_{Jt}(\mathcal{R}(I))$ and $C^{I,J}$ are

$$gr_{Jt}(\mathcal{R}(I))_{(i+j,j)} = \frac{I^i J^j}{I^{i-1} J^{j+1}} t^{i+j} U^j \quad \text{and} \quad C^{I,J}_{(i+j,j)} = \frac{I^i}{JI^{i-1}} t^i [T_1, \dots, T_d]_j,$$

respectively. Notice that we have an R-algebra isomorphism ϕ : $R[T_1, \dots, T_d]/(\{a_i T_j - a_j T_i\}_{i,j}) \cong R[JtU] = \mathcal{R}(J)$ defined by $\phi(\overline{T_i}) = a_i t U$, $i = 1, \dots, d$. Observe that we write tU instead of only t to bear in mind the the bigraduation.

Given a $B-$bigraded module M and an integer $p \in \mathbb{Z}$, we denote by $M_{[p]}$ the additive sub-group of M defined by the direct sum of the pieces $M_{(m,n)}$ such that $m-n = p+1$. Notice that the product by the variable T_i induces an endomorphism of $R[T_1,\dots,T_d]-$modules $M_{[p]} \xrightarrow{T_i} M_{[p]}$, and the product by V_j a morphism of $R[T_1,\dots,T_d]-$modules $M_{[p]} \xrightarrow{V_j} M_{[p+1]}$. Hence $M_{\geq p} = \bigoplus_{n\geq p} M_{[n]}$ is a sub-B-module of M, and we can consider the exact sequence of $R[T_1,\dots,T_d]$-modules

$$0 \longrightarrow M_{[p]} \longrightarrow M_{\geq p} \longrightarrow M_{\geq p+1} \longrightarrow 0.$$

Moreover, in our case, the modules $K^{I,J}_{[p]}$, $C^{I,J}_{[p]}$ and $gr_{Jt}(\mathcal{R}(I))_{[p]}$ are $\mathcal{R}(J)-$modules.

Next lemma shows that $K^{I,J}_{[p]}$, $C^{I,J}_{[p]}$ and $gr_{Jt}(\mathcal{R}(I))_{[p]}$ do not eventually vanish for a finite set of indexes $p \in \mathbb{Z}$.

Lemma 2.1. *(i) For all $p \leq -2$ or $p \geq r_J(I)$, $C^{I,J}_{[p]} = 0$, $gr_{Jt}(\mathcal{R}(I))_{[p]} = 0$ and $K^{I,J}_{[p]} = 0$.*
(ii) π induces the following isomorphisms of $\mathcal{R}(J)-$modules:

$$gr_{Jt}(\mathcal{R}(I))_{[0]} \cong C^{I,J}_{[0]} \cong \frac{I}{J} t[T_1,\dots,T_d],$$

$$gr_{Jt}(\mathcal{R}(I))_{[-1]} \cong \mathcal{R}(J), \quad C^{I,J}_{[-1]} \cong R[T_1,\dots,T_d].$$

Moreover, $K^{I,J}_{[0]} = 0$.

Proof. In order to prove (i), first we observe how $C^{I,J}_{[p]}$ and $gr_{Jt}(\mathcal{R}(I))_{[p]}$ are. Since $C^{I,J} \cong \bigoplus_{i\geq 0} \frac{I^i}{JI^{i-1}} t^i[T_1,\dots,T_d]$, we have that

$$C^{I,J}_{[p]} = \bigoplus_{m-n=p+1} C^{I,J}_{(m,n)} = \frac{I^{p+1}}{JI^p} t^{p+1}[T_1,\dots,T_d],$$

and

$$gr_{Jt}(\mathcal{R}(I))_{[p]} = \bigoplus_{m-n=p+1} gr_{Jt}(\mathcal{R}(I))_{(m,n)} = \bigoplus_{i\geq 0} \frac{J^i I^{p+1}}{J^{i+1} I^p} t^{p+1+i} U^i.$$

Since $I^i = 0$ for all $i < 0$, we have that $C^{I,J}_{[p]} = 0$ and $gr_{Jt}(\mathcal{R}(I))_{[p]} = 0$ for all $p \leq -2$. By the definition of $r_J(I)$ we have $I^{p+1} = JI^p$ for all $p \geq r_J(I)$, so $C^{I,J}_{[p]} = 0$ and $gr_{Jt}(\mathcal{R}(I))_{[p]} = 0$ for all $p \geq r_J(I)$. Notice that we have that $K^{I,J}_{[p]} \subseteq C^{I,J}_{[p]}$ for each $p \in \mathbb{Z}$. Therefore, we have that $K^{I,J}_{[p]} = 0$ for all $p \leq -2$ and $p \geq r_J(I)$.

(ii) When $p=0$ we have

$$C^{I,J}_{[0]} = \frac{I}{J} t[T_1, \dots, T_d] = It(R/J)[T_1, \dots, T_d]$$

and

$$gr_{Jt}(\mathcal{R}(I))_{[0]} = \bigoplus_{i \geq 0} \frac{J^i I}{J^{i+1}} t^{i+1} U^i \cong It gr_J(R).$$

Since $gr_J(R) \cong (R/J)[T_1, \dots, T_d]$, clearly we have that $C^{I,J}_{[0]} \cong gr_{Jt}(\mathcal{R}(I))_{[0]}$. By the exact sequence (1) we deduce $K^{I,J}_{[0]} = 0$. If $p=-1$ then we have $C^{I,J}_{[-1]} = R[T_1, \dots, T_d]$ and $gr_{Jt}(\mathcal{R}(I))_{[-1]} = \bigoplus_{i \geq 0} J^i t^i U^i \cong \mathcal{R}(J)$. $\square$

Let us consider the following bigraded finitely generated B-modules:

$\Sigma^{I,J} := \bigoplus_{p \geq 0} gr_{Jt}(\mathcal{R}(I))_{[p]}$,

$\mathcal{M}^{I,J} := \bigoplus_{p \geq 0} C^{I,J}_{[p]} \cong \bigoplus_{p \geq 0} I^{p+1}/I^p J \; t^{p+1}[T_1, \dots, T_d]$,

and from now on we consider the new

$K^{I,J} := \bigoplus_{p \geq 0} K^{I,J}_{[p]}$.

Notice that by Lemma 2.1 there exists a natural isomorphism of $\mathcal{R}(J)-$modules

$$gr_{Jt}(\mathcal{R}(I)) \cong \mathcal{R}(J) \oplus \Sigma^{I,J}.$$

Since the modules $\Sigma^{I,J}$ and $\mathcal{M}^{I,J}$ are annihilated by J, from the exact sequence (1) we deduce the following exact sequence of $A = B \otimes_R R/J \cong R/J[V_1, \dots, V_\mu; T_1, \dots, T_d]-$bigraded modules

$$0 \longrightarrow K^{I,J} \longrightarrow \mathcal{M}^{I,J} \longrightarrow \Sigma^{I,J} \longrightarrow 0. \tag{S}$$

Notice that from Lemma 2.1 all relevant information of (1) is encoded in the exact sequence (S).

For all $p \geq 0$ we have an exact sequence of $R/J[T_1, \dots, T_d]-$modules

$$0 \longrightarrow K^{I,J}_{[p]} \longrightarrow \mathcal{M}^{I,J}_{[p]} = \frac{I^{p+1}}{JI^p}[T_1, \dots, T_d] \longrightarrow \Sigma^{I,J}_{[p]} \longrightarrow 0, \tag{$\mathrm{S}_{[p]}$}$$

so we can consider the (classic) Hilbert function of $\Sigma^{I,J}_{[p]}$, $\mathcal{M}^{I,J}_{[p]}$ and $K^{I,J}_{[p]}$ with respect the variables $T_1, \dots, T_d$.

Definition 2.2. *Let I be an $\mathbf{m}$-primary ideal and let J be a minimal reduction of I, $\Sigma^{I,J}$ is the bigraded Sally module of I with respect J.*

In the next result we show some properties of the A-module $\mathcal{M}^{I,J}$ that will be used in this paper.

Proposition 2.3. *Given an* $\mathbf{m}$*-primary ideal* I *of* R *with minimal reduction* J*. Then* $Ass_A(\mathcal{M}^{I,J}) = \{\mathbf{m}A + (V_1, \dots, V_\mu)\}$ *and* $\mathcal{M}^{I,J}$ *is a Cohen-Macaulay* $A-$*module of dimension* d*.*

Proof. By the definitions of $Ass_A(\mathcal{M}^{I,J})$ and $\mathcal{M}^{I,J}$, with a simply computation we can show that $Ass_A(\mathcal{M}^{I,J}) = \{Q\}$, where $Q = \mathbf{m}A + (V_1, \dots, V_\mu)$. Similarly, it is easy to see that $Ann_A(\mathcal{M}^{I,J}) = Q$. Notice that $A/Q \cong (R/\mathbf{m})[T_1, \dots, T_d]$. Hence,

$$dim(\mathcal{M}^{I,J}) = dim(A/Ann(\mathcal{M}^{I,J})) = dim(A/Q) = dim((R/\mathbf{m})[T_1, \dots, T_d]) = d.$$

Since $T_1, \dots, T_d$ is a regular sequence in A we have that $\mathcal{M}^{I,J}$ is a Cohen-Macaulay module of dimension d. $\square$

Remark 2.4. Notice that the length as $R-$module of

$$\Sigma^{I,J}_{(m+1,*)} \cong \bigoplus_{j=0}^{j=m} \frac{I^{m+1-j}J^j}{I^{m-j}J^{j+1}} t^{m+1}U^j$$

is equal to the length of

$$S_J(I)_m \oplus \frac{IJ^m}{J^{m+1}},$$

where $S_J(I)_m$ is the degree m piece of the Sally module $S_J(I)$.

On the other hand

$$\Sigma^{I,J}_{[p]} = gr_{Jt}(\mathcal{R}(I))_{[p]} = \bigoplus_{i \geq 0} \frac{J^i I^{p+1}}{J^{i+1}I^p} t^{p+1+i}U^i$$

is isomorphic as $\mathcal{R}(J)-$module to the module L_p defined by Vaz-Pinto in [18] for all $p \geq 1$.

In several papers of Wang appear the modules $T_{k,n}, S_{k,n}$ defined as the kernel and co-kernel of some exact sequence, [19], [20], [21], [22] and [23]. One of the key results of Wang's papers is to prove that there exists the Hilbert function of $T_{k,n}$ and $S_{k,n}$.

In our framework these results follows from the following commutative diagram:

$$\begin{tikzcd}
 & & \bigoplus_{j=1}^{\binom{n+d-1}{d-1}} \frac{I^j}{JI^{j-1}} \arrow[dr] \arrow[dd, "\cong"'] & & \\
0 \arrow[r] & T_{k,n} = K^{I,J}_{(k+n,n)} \arrow[ur] \arrow[dr] & & S_{k,n} = \Sigma^{I,J}_{(k+n,n)} \arrow[r] & 0 \\
 & & \mathcal{M}^{I,J}_{(k+n,n)} \arrow[ur] & &
\end{tikzcd}$$

Finally the $\mathcal{R}(J)-$module $C_p = \bigoplus_{i\geq 1} \frac{I^{i+p}}{J^i I^p}$, defined by Vaz-Pinto, can be linked to $\Sigma^{I,J}_{\geq p}$ by means of the following sets of exact sequences of $\mathcal{R}(J)-$modules, [18]. Here we set $r = r_J(I)$. There exist two sets of exact sequences of $\mathcal{R}(J)-$modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma^{I,J}_{[1]} & \longrightarrow & S_J(I) = C_1 & \longrightarrow & C_2 & \longrightarrow & 0 \\
0 & \longrightarrow & \Sigma^{I,J}_{[2]} & \longrightarrow & C_2 & \longrightarrow & C_3 & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & \Sigma^{I,J}_{[r-2]} & \longrightarrow & C_{r-2} & \longrightarrow & C_{r-1} = \Sigma^{I,J}_{[r-1]} & \longrightarrow & 0
\end{array} \tag{SQ1}$$

and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Sigma^{I,J}_{[0]} = 0 & \longrightarrow & \Sigma^{I,J} = \Sigma^{I,J}_{\geq 0} & \longrightarrow & \Sigma^{I,J}_{\geq 1} & \longrightarrow & 0 \\
0 & \longrightarrow & \Sigma^{I,J}_{[1]} & \longrightarrow & \Sigma^{I,J}_{\geq 1} & \longrightarrow & \Sigma^{I,J}_{\geq 2} & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & \Sigma^{I,J}_{[r-2]} & \longrightarrow & \Sigma^{I,J}_{\geq r-2} & \longrightarrow & \Sigma^{I,J}_{\geq r-1} = \Sigma^{I,J}_{[r-1]} & \longrightarrow & 0
\end{array} \tag{SQ2}$$

Now, given a bigraded $A-$module M, $A = R/J[V_1, \ldots, V_\mu; T_1, \ldots, T_d]$, we can consider the Hilbert function of M defined by

$$h_M(m,n) = \sum_{0\leq j\leq n} length_A(M_{(m,j)})$$

and by Theorem 1.2, there exist integers $f_{i,j}(M) \in \mathbb{Z}$, $i \geq 0$, $j \geq 0$, and $i+j \leq c-1$, for some integer $c \geq 0$, such that the polynomial

$$p_M(m,n) = \sum_{i+j\leq c-1} f_{i,j}(M) \binom{m}{i} \binom{n}{j}$$

verifies $p_M(m,n) = h_M(m,n)$ for all $m \geq m_0$ and $n \geq n_0 + m$ for some integers $m_0, n_0 \geq 0$.

Lemma 2.5. *Let M be a bigraded $A-$module for which there exist integers $a \leq b$ such that $M_{[p]} = 0$ for all $p \notin [a,b]$, then $f_{i,j}(M) = 0$ for all $j \geq 1$.*

Proof. Since $M_{[p]} = \bigoplus_{m-n=p+1} M_{(m,n)}$. If $M_{[p]} = 0$ for all $p \notin [a,b]$, we have that $M_{(m,n)} = 0$ for all pair (m,n) such that $m - n > b+1$ or $m - n < a+1$. If we take $m \geq m_0$, $n \geq n_0 + m$ we have

$$h_M(m,n) = p_M(m,n) = \sum_{i+j \leq c-1} f_{i,j}(M) \binom{m}{i} \binom{n}{j}$$

We can suppose that $m \geq b+1$ and $n \geq m - b - 1$, so we have

$$\begin{aligned} h_M(m,n) &= \sum_{0 \leq j \leq n} length_A(M_{(m,j)}) \\ &= \sum_{m-b-1 \leq j \leq m-a-1} length_A(M_{(m,j)}) = \sum_{j \in \mathbb{Z}} length_A(M_{(m,j)}) \end{aligned}$$

because $M_{(m,j)} = 0$ when $j < m - b - 1$ and $j > m - a - 1$. Hence, for m, n large enough we have that

$$h_M(m,n) = p_M(m,n) = h_M(m) = p_M(m).$$

Therefore

$$\sum_{i+j \leq c-1} f_{i,j}(M) \binom{m}{i} \binom{n}{j} = \sum_i a_i \binom{m}{i}.$$

Since $\{\binom{m}{i}\binom{n}{j}\}_{i,j}$ is a basis of the polynomial ring in m and n, we have that $f_{i,j}(M) = 0$ for all $j \geq 1$. $\square$

Given an $A-$module under the hypothesis of the above Lemma we can write

$$p_M(m,n) = p_M(m) = \sum_{i=0}^{c-1} f_{i,0}(M) \binom{m+c-i}{c-i},$$

and $h_M(m,n) = p_M(m)$ for $m \geq m_0$ and $n \geq n_0 + m$. From Lemma 2.1 we can apply the last result to the $A-$modules $\Sigma^{I,J}$, $\mathcal{M}^{I,J}$, and $K^{I,J}$.

Proposition 2.6. *Let I be an $\mathbf{m}-$primary ideal of R with minimal reduction J. Then*

$$p_{\Sigma^{I,J}}(m) = \sum_{i=0}^{d-1} (-1)^i e_{i+1}(I) \binom{m-1+d-i-1}{d-i-1}$$

Proof. Since the length of $\Sigma^{I,J}_{(m+1,*)}$ is equal to the length of $S_J(I)_m \oplus \frac{IJ^m}{J^{m+1}}$ as $R-$modules, we have that

$$h_{\Sigma^{I,J}}(m,n) = length_R(S_{m-1}) + length_R(\frac{IJ^{m-1}}{J^m})$$

for all $m \geq m_0$, $n \geq n_0 + m$. Since $gr_J(R) \cong (R/J)[T_1, \dots, T_d]$, clearly we get that $Igr_J(R) \cong (I/J)[T_1, \dots, T_d]$. Thus, $length_R(\frac{IJ^{m-1}}{J^m})$ coincides with the length of the piece of degree $m-1$ of $(I/J)[T_1, \dots, T_d]$. So we have

$$length_R(\frac{IJ^{m-1}}{J^m}) = length_R(I/J) \binom{m-1+d-1}{d-1}.$$

Hence we deduce that, Lemma 2.5,

$$p_{\Sigma^{I,J}}(m) = p_{S_J(I)}(m-1) + length_R(I/J) \binom{m-1+d-1}{d-1}.$$

Let us recall that there exist integers $s_0, \dots, s_{d-1}$ such that

$$p_{S_J(I)}(n) = \sum_{i=0}^{d-1} (-1)^i s_i \binom{n+d-i-1}{d-i-1}$$

and that $s_0 = e_1(I) - length_R(I/J)$, $s_i = e_{i+1}(I)$ for $i = 1, \dots, d-1$, [17]. Hence, we have

$$p_{S_J(I)}(m-1) = \sum_{i=0}^{d-1} (-1)^i e_{i+1}(I) \binom{m-1+d-i-1}{d-i-1} - length_R(I/J) \binom{m-1+d-1}{d-1}$$

and then

$$\begin{aligned} p_{\Sigma^{I,J}}(m) &= p_{S_J(I)}(m-1) + length_R(I/J) \binom{m-1+d-1}{d-1} \\ &= \sum_{i=0}^{d-1} (-1)^i e_{i+1}(I) \binom{m-1+d-i-1}{d-i-1} \end{aligned}$$

□

In the next proposition we compute the multiplicities of the modules $\mathcal{M}^{I,J}$, $\Sigma^{I,J}$ and $K^{I,J}$. Notice that they are related with the integer $\Lambda(I,J)$ defined in the introduction. From now on we consider the integer $\delta(I,J) = \Lambda(I,J) - e_1(I)$.

Proposition 2.7. *The following conditions hold:*
(i) $deg(p_{\mathcal{M}^{I,J}}) = d-1$ *and* $e_0(\mathcal{M}^{I,J}) = \Lambda(I,J)$.
(ii) If $\Sigma^{I,J} = 0$ *then* $gr_I(R)$ *is a Cohen-Macaulay ring . If* $\Sigma^{I,J} \neq 0$ *then* $deg(p_{\Sigma^{I,J}}) = d-1$ *and* $e_0(\Sigma^{I,J}) = e_1(I)$.
(iii) $e_0(K^{I,J}) = \delta(I,J)$*; if* $K^{I,J} \neq 0$ *then* $deg(p_{K^{I,J}}) = d-1$*. In particular,*

$$\Lambda(I,J) \geq e_1(I).$$

Proof. (i) We know that $\mathcal{M}^{I,J}$ is Cohen-Macaulay of dimension d, Proposition 2.3. By Lemma 2.1 and Lemma 2.5 we have that $p_{\mathcal{M}^{I,J}}(m,n) = p_{\mathcal{M}^{I,J}}(m)$, so $deg(p_{\mathcal{M}^{I,J}}) = d-1$. Since $\mathcal{M}^{I,J} \cong \bigoplus_{p\geq 0} I^{p+1}/I^pJ\ t^{p+1}[T_1,\dots,T_d]$ we have that

$$\begin{aligned} h_{\mathcal{M}^{I,J}}(m,n) &= \sum_{i=0}^{n} length_R\left(\mathcal{M}^{I,J}_{(m,i)}\right) = \sum_{i=0}^{n} length_R\left(\frac{I^{m-i}}{I^{m-i-1}J}[T_1,\dots,T_d]_i\right) \\ &= length_R\left(\frac{I^m}{I^{m-1}J}[T_1,\dots,T_d]_0\right) + length_R\left(\frac{I^{m-1}}{I^{m-2}J}[T_1,\dots,T_d]_1\right) \\ &\quad + \cdots + length_R\left(\frac{I^{m-n}}{I^{m-n-1}J}[T_1,\dots,T_d]_n\right) \end{aligned}$$

Notice that for $n \geq m \gg 0$ we have

$$\begin{aligned} p_{\mathcal{M}^{I,J}}(m,n) = h_{\mathcal{M}^{I,J}}(m,n) &= length_R\left(\frac{I^m}{I^{m-1}J}[T_1,\dots,T_d]_0\right) + \\ &\quad + length_R\left(\frac{I^{m-1}}{I^{m-2}J}[T_1,\dots,T_d]_1\right) + \\ &\quad + \cdots + length_R\left(\frac{I}{J}[T_1,\dots,T_d]_{m-1}\right) \end{aligned}$$

Moreover, for $m \geq r_J(I)$ we have

$$\begin{aligned} p_{\mathcal{M}^{I,J}}(m,n) &= \sum_{i\geq 1} length_R\left(\frac{I^i}{I^{i-1}J}[T_1,\dots,T_d]_{m-i}\right) \\ &= \sum_{i\geq 1} length_R\left(\frac{I^i}{I^{i-1}J}\right)\binom{m-i+d-1}{d-1} \end{aligned}$$

Each binomial number is a polynomial in m of degree $d-1$, and the leading coefficient of this polynomial $p_{\mathcal{M}^{I,J}}$ gives us the multiplicity

$$e_0(\mathcal{M}^{I,J}) = \sum_{i\geq 1} length_R\left(\frac{I^i}{I^{i-1}J}\right) = \Lambda(I,J)$$

(ii) If $\Sigma^{I,J} = 0$ then $S_J(I) = 0$ and $gr_I(R)$ is Cohen-Macaulay, [18] Remark 2.4. Let us assume that $\Sigma^{I,J} \neq 0$. Since, Proposition 2.6,

$$p_{\Sigma^{I,J}}(m) = \sum_{i=0}^{d-1} (-1)^i e_{i+1}(I) \binom{m-1+d-i-1}{d-i-1},$$

then $deg(p_{\Sigma^{I,J}}) = d-1$ and $e_0(\Sigma^{I,J}) = e_1(I)$.
(iii) From the additivity of the multiplicity in (S) and by (i) and (ii) we get

$$e_0(K^{I,J}) = e_0(\mathcal{M}^{I,J}) - e_0(\Sigma^{I,J}) = \Lambda(I,J) - e_1(I).$$

If $K^{I,J} \neq 0$ then $deg(p_{K^{I,J}}) = d-1$. Notice that $deg(p_{\Sigma^{I,J}}) = deg(p_{\mathcal{M}^{I,J}}) = d-1$, so $e_0(K^{I,J}) \geq 0$ and then $\Lambda(I,J) \geq e_1(I)$. $\square$

Proposition 2.8. *(i) For all* $p \geq 0$

$$e_0(\Sigma^{I,J}_{[p]}) = length_R(I^{p+1}/JI^p) - e_0(K^{I,J}_{[p]}) \geq 0,$$

and

$$e_1(I) = \sum_{p\geq 0} (length_R(I^{p+1}/JI^p) - e_0(K^{I,J}_{[p]})).$$

(ii) For all $p \geq 0$

$$length_R(I^{p+1} \cap J/JI^p) \geq e_0(K^{I,J}_{[p]}),$$

and

$$\delta(I,J) = e_0(K^{I,J}) = \sum_{p\geq 0} e_0(K^{I,J}_{[p]}) \geq 0.$$

Proof. (i) From Proposition 2.7 we have

$$e_1(I) = e_0(\Sigma^{I,J}) = \sum_{p\geq 0}^{*} e_0(\Sigma^{I,J}_{[p]}),$$

here $*$ stands for the integers p such that $deg(p_{\Sigma^{I,J}_{[p]}}) = d-1$.

From the exact sequence of R-modules

$$0 \longrightarrow K^{I,J}_{[p]} \longrightarrow \frac{I^{p+1}}{JI^p}[T_1,\dots,T_d] \longrightarrow \Sigma^{I,J}_{[p]} \longrightarrow 0,$$

we deduce that if $deg(p_{\Sigma^{I,J}_{[p]}}) < d-1$ then $length_R(I^{p+1}/JI^p) = e_0(K^{I,J}_{[p]})$. Let us assume $deg(p_{\Sigma^{I,J}_{[p]}}) = d-1$, from the additivity of the multiplicity we deduce

$$e_0(\Sigma^{I,J}_{[p]}) = length_R(I^{p+1}/JI^p) - e_0(K^{I,J}_{[p]}),$$

so
$$e_1(I) = e_0(\Sigma^{I,J}) = \sum_{p\geq 0}{}^{*} e_0(\Sigma^{I,J}_{[p]}) = \sum_{p\geq 0}(length_R(I^{p+1}/JI^p) - e_0(K^{I,J}_{[p]})).$$

(ii) From the well known result $gr_J(R) \cong R/J[T_1,\dots,T_d]$ it is easy to prove that $K^{I,J}_{[p]}$ is in fact a submodule of
$$\frac{I^{p+1}\cap J}{JI^p}[T_1,\dots,T_d].$$

From this we deduce
$$length_R(I^{p+1}\cap J/JI^p) \geq e_0(K^{I,J}_{[p]}).$$

The second part of the claim follows from Proposition 2.7 (iii)
$$\delta(I,J) = \Lambda(I,J) - e_1(I) = e_0(K^{I,J}) = \sum_{p\geq 0} e_0(K^{I,J}_{[p]}) \geq 0.$$

□

Remark 2.9. Notice that some of the results on $T_{k,n} = K^{I,J}_{(k+n,n)}$ of [19], [20], [22], [21] and [23] are corollaries of Proposition 2.7 and Proposition 2.8.

Let us recall that from [21] and [11] we have
$$\Delta(I,J) \geq \delta(I,J) = \Lambda(I,J) - e_1(I) \geq 0.$$

In the next result we show that these inequalities can we deduced from some "local" inequalities. For all $p \geq 0$ we define the following the integers

$\Delta_p(I,J) = length_R(I^{p+1}\cap J/JI^p)$,

$\delta_p(I,J) = e_0(K^{I,J}_{[p]})$, and

$\Lambda_p(I,J) = length_R(I^{p+1}/JI^p)$.

From the last result we deduce:

Proposition 2.10. *For all $p \geq 0$ the following inequalities hold*
$$\Delta_p(I,J) \geq \delta_p(I,J) = \Lambda_p(I,J) - e_0(\Sigma^{I,J}_{[p]}) \geq 0.$$

Summing up these inequalities with respect p we get
$$\Delta(I,J) \geq \delta(I,J) = \Lambda(I,J) - e_1(I) \geq 0.$$

3. On the depth of the blow-up algebras

The aim of this section is to prove a refined version of Wang's conjecture by considering some special configurations of the set $\{\delta_p(I,J)\}_{p\geq 0}$ instead of $\delta = \sum_{p\geq 0}\delta_p(I,J)$, Theorem 3.3. As a by-product we recover the known cases of Wang's conjecture, Corollary 3.6, and we prove a weak version of Sally's conjecture, Corollary 3.7.

Inspired by the proof of Polini, in the next result we generalize Claim 3 of [13].

Theorem 3.1. *Assume that $d \geq 3$. Let I be an $\mathbf{m}-$primary ideal of R and let J a minimal reduction of I. Let us assume that $K^{I,J} \neq 0$, and either $K^{I,J}_{[p]} = 0$ or $K^{I,J}_{[p]}$ is a rank one torsion free $\mathbf{k}[T_1,\dots,T_d]$-module for $p \geq 0$. Then*

$$depth(gr_{Jt}(\mathcal{R}(I))) \geq d-1.$$

Proof. Let $p_1 < \ldots < p_n$ be the sequence of integers such that $K^{I,J}_{[p_i]} \neq 0$, $i = 1,\dots,n$. By Lemma 2.1 we have a finite number of these integers. Hence, from the sequence $(\mathrm{S}_{[p]})$ we get

$$\Sigma^{I,J}_{[p]} \cong \mathcal{M}^{I,J}_{[p]} = \frac{I^{p+1}}{JI^p}[T_1,\dots,T_d]$$

for $p \neq p_1,\dots,p_n$, and the following exact sequences of $R-$modules

$$0 \longrightarrow K^{I,J}_{[p_i]} \longrightarrow \frac{I^{p_i+1}}{JI^{p_i}}[T_1,\dots,T_d] \longrightarrow \Sigma^{I,J}_{[p_i]} \longrightarrow 0, \qquad (2)$$

$i = 1,\dots,n$. Notice that by hypothesis $K^{I,J}_{[p_i]}$ is isomorphic to an ideal I_i of $D = \mathbf{k}[T_1,\dots,T_d]$, $i = 1,\dots,n$.

Let $\mathfrak{p}$ be a height $h \geq 2$ prime ideal of D. Since $D = \mathcal{R}(J)/\mathbf{m}\mathcal{R}(J)$, there exists a prime ideal $\mathfrak{q}$ of $\mathcal{R}(J)$ such that $\mathfrak{p} = \mathfrak{q}/\mathbf{m}\mathcal{R}(J)$.

Since $depth_{\mathfrak{q}}(S_J(I)) \geq 1$, [13], by depth counting in the set of exact sequences of $\mathcal{R}(J)-$modules (SQ1) we get that $depth_{\mathfrak{q}}(\Sigma^{I,J}_{[p_1]}) \geq 1$. In fact, for $i < p_1$ we have that $\Sigma^{I,J}_{[i]} \cong \mathcal{M}^{I,J}_{[i]}$, so $depth_{\mathfrak{q}}(\Sigma^{I,J}_{[i]}) \geq d > 2$. Thus, if $p_1 \neq 1$ then

$$depth_{\mathfrak{q}}(C_2) \geq min\{depth_{\mathfrak{q}}(C_1), depth_{\mathfrak{q}}(\Sigma^{I,J}_{[1]}) - 1\} \geq 1.$$

Hence, while $i < p_1$ we have that $depth_{\mathfrak{q}}(C_{i+1}) \geq 1$, and it implies that

$$depth_{\mathfrak{q}}(\Sigma^{I,J}_{[p_1]}) \geq min\{depth_{\mathfrak{q}}(C_{p_1}), depth_{\mathfrak{q}}(C_{p_1+1}) + 1\} \geq 1.$$

Otherwise, if $p_1 = 1$ then

$$depth_{\mathfrak{q}}(\Sigma^{I,J}_{[1]}) \geq min\{depth_{\mathfrak{q}}(C_1), depth_{\mathfrak{q}}(C_2) + 1\} \geq 1.$$

Hence we have $depth_{\mathfrak{q}}(\Sigma^{I,J}_{[p_1]}) \geq 1$.

Depth counting on (2) yields

$$depth_{\mathfrak{q}}(K^{I,J}_{[p_1]}) \geq min\{depth_{\mathfrak{q}}(\frac{I^{p_1+1}}{JI^{p_1}}[T_1,\dots,T_d]), depth_{\mathfrak{q}}(\Sigma^{I,J}_{[p_1]})+1\} \geq 2$$

because $depth_{\mathfrak{q}}(\frac{I^{p_1+1}}{JI^{p_1}}[T_1,\dots,T_d]) \geq d$ and $depth_{\mathfrak{q}}(\Sigma^{I,J}_{[p_1]}) \geq 1$. Then

$$depth(I_1)_{\mathfrak{p}} = depth(I_1)_{\mathfrak{q}} \geq depth_{\mathfrak{q}}(K^{I,J}_{[p_1]}) \geq 2,$$

[12]. In particular we have that $\mathfrak{p} \notin Ass_D(I_1)$, because $depth_{\mathfrak{p}}(I_1) \geq 2$, so I_1 is an unmixed ideal of D of height one. In fact, all the associated primes of I_1 have height ≤ 1, and being D a domain and $I_1 \neq 0$, the associated ideals of I_1 are height 1 ideals. Since D is factorial we deduce that $I_1 \subset D$ is principal, and then $depth(K^{I,J}_{[p_1]}) = d$.

Since $depth_{\mathfrak{q}}(\frac{I^{p_1+1}}{JI^{p_1}}[T_1,\dots,T_d]) \geq d$ and $depth_{\mathfrak{q}}(K^{I,J}_{[p_1]}) = d$, by depth counting on (2), we deduce that $depth_{\mathfrak{q}}(\Sigma^{I,J}_{[p_1]}) \geq d-1 \geq 2$.

By depth counting on (SQ1) we get that $depth_{\mathfrak{q}}(C_{p_1+1}) \geq 1$. In fact,

$$depth_{\mathfrak{q}}(C_{p_1+1}) \geq min\{depth_{\mathfrak{q}}(C_{p_1}), depth_{\mathfrak{q}}(\Sigma^{I,J}_{[p_1]})-1\} \geq 1,$$

since $depth_{\mathfrak{q}}(C_{p_1}) \geq 1$. We can iterate the process and we get that $depth_{\mathfrak{q}}(\Sigma^{I,J}_{[p]}) \geq d-1$ for all p, in particular we get that the $\mathcal{R}(J)-$module $\Sigma^{I,J}$ verifies

$$depth(\Sigma^{I,J}_{[p]}) \geq d-1.$$

From the last row of the sequence (SQ2) we get that

$$depth(\Sigma^{I,J}_{\geq r-2}) \geq min\{depth(\Sigma^{I,J}_{[r-2]}), depth(\Sigma^{I,J}_{[r-1]})\} \geq d-1.$$

Iterating this process in (SQ2) we deduce that

$$depth(\Sigma^{I,J}) \geq d-1.$$

Now, let consider the exact sequence of $\mathcal{R}(J)-$modules

$$0 \longrightarrow \mathcal{R}(J) \longrightarrow gr_{Jt}(\mathcal{R}(I)) \longrightarrow \Sigma^{I,J} \longrightarrow 0,$$

depth counting in this sequence give us the claim, because

$$depth(gr_{Jt}(\mathcal{R}(I))) \geq min\{depth(\mathcal{R}(J)), depth(\Sigma^{I,J})\}$$

with $depth(\mathcal{R}(J)) = d+1$ and $depth(\Sigma^{I,J}) \geq d-1$. □

Lemma 3.2. *If $\delta_p(I,J) = 1$ then $K^{I,J}_{[p]}$ is a rank one torsion free $\mathbf{k}[T_1,\dots,T_d]$-module.*

Proof. Let us recall that

$$K_{[p]}^{I,J} \subset \mathcal{M}_{[p]}^{I,J} = \frac{I^{p+1}}{JI^p}[T_1, \dots, T_d]$$

We denote by $K_{[p],j}^{I,J}$ the homogeneous piece of degree j of $K_{[p]}^{I,J}$ with respect $T_1, \dots, T_d$. Since $\frac{I^{p+1}}{JI^p}$ is a finite length R-module there exists a composition series

$$0 = N_l \subset N_{l-1} \subset \cdots \subset N_0 = \frac{I^{p+1}}{JI^p}$$

such that N_i is a sub-R-module of I^{p+1}/JI^p and $N_i/N_{i+1} \cong \mathbf{k}$ for all $i = 0, \dots, l-1$, i.e. $length_R(I^{p+1}/JI^p) = l$. Hence we have a sequence of $R[T_1, \dots, T_d]-$modules

$$0 = N_l[T_1, \dots, T_d] \subset N_{l-1}[T_1, \dots, T_d] \subset \cdots \subset N_0[T_1, \dots, T_d] = \frac{I^{p+1}}{JI^p}[T_1, \dots, T_d].$$

If we denote by

$$W_i = \frac{K_{[p]}^{I,J} \cap N_i[T_1, \dots, T_d]}{K_{[p]}^{I,J} \cap N_{i+1}[T_1, \dots, T_d]} \subset \frac{N_i[T_1, \dots, T_d]}{N_{i+1}[T_1, \dots, T_d]} = \mathbf{k}[T_1, \dots, T_d]$$

Since $e_0(K_{[p]}^{I,J}) = 1$ we have that $K_{[p]}^{I,J} \neq 0$. If $W_i = 0$ for all $i = 0, \dots, l$ then $K_{[p]}^{I,J} = 0$, so there exists a set of indexes $0 \le i_1 \le \ldots \le i_s \le l$ such that $W_{i_j} \neq 0$, $j = 1, \dots, s$.

We denote by $W_{i,m}$ the degree m piece of W_i with respect $T_1, \dots, T_d$. Let m_0 be an integer such that for all $j = 1, \dots, s$ we have $W_{i_j,m_0} \neq 0$. This integer exists. In fact, for each $W_{i_j} \neq 0$, there exists an integer m_{i_j} such that $W_{i_j,m_{i_j}} \neq 0$. Then, for any $t \ge 0$ we have that $W_{i_j,m_{i_j}+t} \neq 0$. So, we can choose the maximum of these $m_{i_1}, \dots, m_{i_s}$. Then we have

$$\begin{aligned} length_R(K_{[p],m}^{I,J}) &= \sum_{j=1}^{s} length_R(W_{i_j,m}) = \sum_{j=1}^{s} length_R(W_{i_j,m_0} \cdot W_{i_j,m-m_0}) \\ &\ge \sum_{j=1}^{s} length_R(W_{i_j,m-m_0}) \\ &\ge s\binom{m-m_0+d-1}{d-1} \end{aligned}$$

for all $m \ge m_0$.

Since $e_0(K_{[p]}^{I,J}) = 1$ we deduce that $s = 1$, because $deg(p_{K_{[p]}^{I,J}}) = d-1$ and the binomial number is a polynomial of the same degree. Hence, $W_i = 0$ for all $i \neq i_1$. Therefore, $K_{[p]}^{I,J} \cap N_i[T_1, \dots, T_d] = 0$ for $i = i_1 + 1, \dots, l$ and $K_{[p]}^{I,J} \cap N_i[T_1, \dots, T_d] =$

$K^{I,J}_{[p]}$ for $i = 0, \dots, i_1$. From this we get that $K^{I,J}_{[p]} = W_{i_1} \subset \mathbf{k}[T_1, \dots, T_d]$, so $K^{I,J}_{[p]}$ is a rank one torsion free $\mathbf{k}[T_1, \dots, T_d]$-module. □

Next theorem is the main result of this paper. We prove a refined version of Wang's conjecture by considering some special configurations of the set $\{\delta_p(I,J)\}_{p\geq 0}$ instead of $\delta = \sum_{p\geq 0} \delta_p(I,J)$. Let us consider $\bar{\delta}(I,J)$ the maximum of the integers $\delta_p(I,J)$ for $p \geq 0$.

Theorem 3.3. *Assume that* $\bar{\delta}(I,J) \leq 1$. *Then*

$$depth(\mathcal{R}(I)) \geq d - \bar{\delta}(I,J)$$

and $depth(gr_I(R)) \geq d - 1 - \bar{\delta}(I,J)$.

Proof. If $\bar{\delta}(I,J) = 0$ then $K^{I,J} = 0$, Proposition 2.7. The exact sequence (S) shows that $\mathcal{M}^{I,J} \cong \Sigma^{I,J}$ as $A-$modules, so $depth(\Sigma^{I,J}) = d$, Proposition 2.3. Let us consider the exact sequence of $A-$modules

$$0 \longrightarrow \Sigma^{I,J} \longrightarrow gr_{Jt}(\mathcal{R}(I)) \longrightarrow \mathcal{R}(J) \longrightarrow 0$$

Depth counting shows that $depth(gr_{Jt}(\mathcal{R}(I))) \geq d$, because $depth(\mathcal{R}(J)) = d+1$ and $depth(\Sigma^{I,J}) = d$. Hence $depth(\mathcal{R}(I)) \geq d$. If $gr_I(R)$ is a Cohen-Macaulay ring then $depth(gr_I(R)) = d \geq d-1$. Otherwise, when $depth(gr_I(R)) < d = depth(R)$, from [10],

$$depth(gr_I(R)) = depth(\mathcal{R}(I)) - 1 \geq d - 1$$

and we get the claim.

Suppose that $\bar{\delta}(I,J) = 1$. For cases $d = 1, 2$ we have the claim easily. So we assume $d \geq 3$. From Lemma 3.2 and Theorem 3.1 we get $depth(gr_{Jt}(\mathcal{R}(I))) \geq d-1$, so $depth(\mathcal{R}(I)) \geq d-1$. Now, if $gr_I(R)$ is Cohen-Macaulay, then $depth(gr_I(R)) = d \geq d-2$. Otherwise, if $depth(gr_I(R)) < d = depth(R)$ then, [10],

$$depth(gr_I(R)) = depth(\mathcal{R}(I)) - 1 \geq (d-1) - 1 = d-2$$

and we get the claim. □

Remark 3.4. The example of Wang in [23], Example 3.13, shows that the last result is sharp in the sense that we cannot expect to have $depth(gr_I(R)) \geq d-1$ provided $\bar{\delta}(I,J) = 1$. Precisely, this is a counterexample for the question formulated by Guerrieri in [5]. She asked if it were true that $depth(gr_I(R)) \geq d-1$ for an $\mathbf{m}-$primary ideal I in a $d-$dimensional Cohen-Macaulay ring provided that

$\Delta_p(I,J) \leq 1$ $\forall p \geq 1$. Wang reformulate the question in the regular case. Relating to this, we are able to improve the bound for the Cohen-Macaulay case:

Proposition 3.5. *Let I an $\mathbf{m}-$primary ideal and J a minimal reduction of I in a Cohen-Macaulay $d-$dimensional local ring. Assuming that $\Delta_p(I,J) \leq 1$ for all $p \geq 1$ we have that*

$$depth(gr_I(R)) \geq d-2.$$

Proof. Observe that $\delta_p(I,J) \leq \Delta_p(I,J) \leq 1$ for all $p \geq 1$. For $p=0$ we have that $\delta_0(I,J) = e_0(K_{[0]}^{I,J}) = 0$ for Lemma 2.1 and use Theorem 3.3. □

In the following corollary, we are able to prove the Conjecture of Wang in the known cases, [21], using the previous results and the bigraded modules defined before. Notice that in general we have

$$\delta(I,J) \geq \bar{\delta}(I,J),$$

so from the last result we deduce:

Corollary 3.6. *Let I be an $\mathbf{m}$-primary ideal and J a minimal reduction of I. Then it holds*

$$depth(gr_I(R)) \geq d-1-\delta(I,J)$$

for $\delta(I,J) = 0,1$.

Proof. Notice that $\delta(I,J) \geq \bar{\delta}(I,J)$. □

Let us recall that Valabrega and Valla characterized under which conditions $gr_I(R)$ is Cohen-Macaulay. They proved that given a minimal reduction J of I then $gr_I(R)$ is Cohen-Macaulay if and only if for all $n \geq 0$ the $n-$th Valabrega-Valla's condition holds, [16]:

$$I^n \cap J = I^{n-1} J,$$

i.e. $\Delta_{n-1}(I,J) = 0$.

In the next result we prove a weak version of Sally's conjecture, [3], [4], [15].

Corollary 3.7. . *Let I be an* $\mathbf{m}-$*primary ideal of R with minimal reduction J. If $I^n \cap J = I^{n-1}J$ for $n = 2, \ldots, t$ and length$(\frac{I^{t+1}}{JI^t}) = \epsilon \leq$ Min$\{1, d-1\}$ then it holds*

$$d - 1 - \epsilon \leq depth(gr_I(R)) \leq d.$$

Proof. From Proposition 2.10 we get that $\delta_p(I, J) = 0$ for all $p \leq t - 1$.

Let us consider the finitely generated R/J-algebra $\mathcal{B}$

$$\mathcal{B} = \frac{\mathcal{R}(I)}{J + Jt\mathcal{R}(I)} = \frac{R}{J} \oplus \bigoplus_{n\geq 1} \frac{I^n}{JI^{n-1}} t^n.$$

Notice that $\mathcal{B}_{\geq 1}$ is the positive part of the degree zero piece with respect U of $\Sigma^{I,J}$. We can consider the Hilbert function of $\mathcal{B}$, $n \geq 0$,

$$h_{\mathcal{B}}(n) = length_{R/J}(I^n/JI^{n-1}).$$

From [1] and [2] we deduce that

$$h_{\mathcal{B}}(t + 1 + n) = length(\frac{I^{t+1+n}}{JI^{t+n}}) \leq \epsilon \leq 1$$

for all $n \geq 0$, so $\delta_p(I, J) \leq 1$ for all $p \geq t$, Proposition 2.10.

We know that $\delta_p(I, J) \leq 1$ for all $p \geq 0$, from Theorem 3.3 we get the claim. □

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